Riemann integrability

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<u>Defn</u>. A subset $S \subset \mathbb{R}$ is a SET OF MEASURE ZERO if for every $\varepsilon > 0$ there exists a cover of S by a sequence of open intervals $(\alpha_i, \beta_i) \subset \mathbb{R}$ $i = 1, 2, 3, \dots$

$$S \subset \bigcup_{i=1}^{\infty} (\alpha_i, \beta_i)$$

such that

$$\sum_{i=1}^{\infty} (\beta_i - \alpha_i) < \varepsilon .$$

Remark: The collection of intervals may also be <u>finite</u>. In ptic, finite sets have measure zero.

<u>Theorem</u>. $f: [a,b] \to \mathbb{R}$, f bounded, and

$$S = \{ x \in [a,b] \mid f \text{ discontinuous at } x \}$$

If S is a set of measure zero, then f is Riemann integrable.

<u>REMARK</u>. In ptic, every continuous function is integrable, as well as every fn. with finitely many discontinuities.

<u>PROOF</u>. Introduce the function $\omega: [a,b] \to \mathbb{R}$

$$\omega(x, \delta) = \sup \{ |f(y) - f(z)| \mid y, z \in (x-\delta, x+\delta) \}$$

$$\omega(x) = \inf \{ \omega(x, \delta) \mid \delta > 0 \}$$

"the oscillation of f at x"

<u>Note</u>: f cont. at x_0

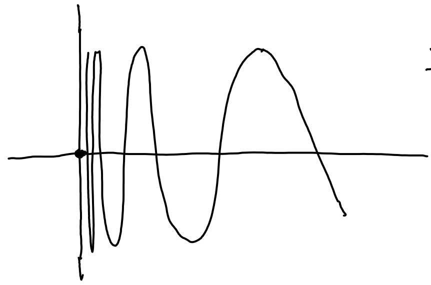
$\Rightarrow$ for every $\varepsilon > 0$ exists $\delta > 0$ s.t. if $|x - x_0| < \delta$ then $|f(x_0) - f(x)| < \varepsilon$.

$\Rightarrow$ for every $\varepsilon > 0$ exists $\delta > 0$ s.t. $\omega(x_0, \delta) < 2\varepsilon$

$\Rightarrow$ $\omega(x_0) = 0$.

Converse is proven in similar fashion.

Example :



$$f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

then $\omega(0) = 2$.

Now fix $\varepsilon > 0$. Choose $r > 0$ such that

$$r < \frac{\varepsilon}{2(b-a)}.$$

Consider the set of points x where f is discontinuous with oscillation $\geq r$,

$$S_r = \{ x \in [a,b] \mid \omega(x) \geq r \}.$$

Claim. The set S_r is closed.

Proof of the claim: suppose x_0 is a cluster point of S_r. Then for every $\delta > 0$ there is a point $y \in S_r$ with $|x_0 - y| < \delta$.

Take $\delta' = \delta - |x_0 - y|$, then

$$r \leq \omega(y) \leq \omega(y, \delta') \leq \omega(x, \delta)$$

because $(y-\delta', y+\delta') \subset (x-\delta, x+\delta)$

Therefore $\omega(x) \geq r$, so $x \in S_r$.

This proves the claim.

S_r is closed and bounded $\Rightarrow$ compact.

By hypothesis, we can cover S and therefore also $S_r \subset S$ by open intervals

$$S_r \subset \bigcup_{i=1}^{\infty} (\alpha_i, \beta_i)$$

with $$\sum_{i=1}^{\infty} (\beta_i - \alpha_i) < \frac{\varepsilon}{2(B-A)}$$

Here $$B = \text{l.u.b.} \{ f(x) \mid x \in [a,b] \}$$
$$A = \text{g.l.b.} \{ f(x) \mid x \in [a,b] \}$$

Since S_r is compact, a finite subset suffices

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$$S_r \subset V = (\alpha_{n_1}, \beta_{n_1}) \cup \cdots \cup (\alpha_{n_k}, \beta_{n_k}).$$

wlog: we can assume these are disjoint.

The complement $[a,b] \setminus V = V^c$ is closed.
For point $x \in V^c$ we have $\omega(x) < r$.
Because

$$\omega(x) = \inf \{ \omega(x,\delta) \mid \delta > 0 \}$$

it follows that for some $\delta > 0$,

$$\omega(x) \leq \omega(x,\delta) < r.$$

So now if $y, z \in (x-\delta, x+\delta)$ then $|f(y) - f(z)| < r$.
We write $I_x = (x - \frac{\delta}{2}, x + \frac{\delta}{2})$

V^c is closed and bounded $\Rightarrow$ compact.
The union $\bigcup_{x \in V} I_x$ covers V^c, and so we can choose a finite collection of I_x that cover V^c.

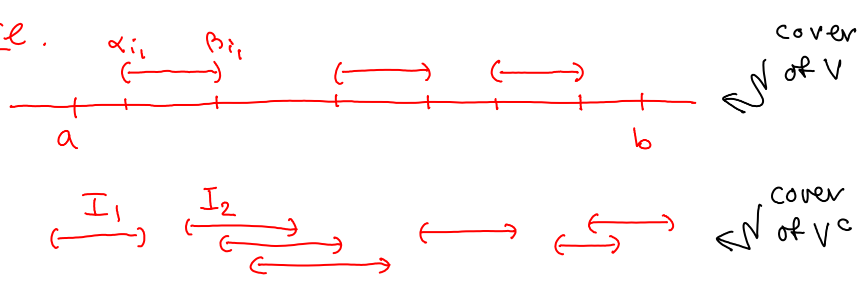
$$V^c \subset I_1 \cup I_2 \cup \cdots \cup I_\ell .$$

NOT disjoint in general

We now choose a partition of $[a,b]$ as follows.
Take the set of all endpoints $\{\alpha_{n_1}, \beta_{n_1}, \ldots, \alpha_{n_k}, \beta_{n_k}\}$ and all endpoints of $I_1, \ldots, I_\ell$ and order them in increasing order. Ignore points in this set if they fall outside the interval $[a,b]$, and add $\{a,b\}$ themselves. The result will be our partition

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b.$$

Picture.



There are two cases.

① $(x_i, x_{i-1}) \subset V$. The total size of these intervals in the partition is very small $< \frac{\varepsilon}{2} \cdot \frac{1}{B-A}$

② $(x_{i-1}, x_i) \subset V^c$. In this case we have $(x_{i-1}, x_i) \subset I_x$ and consequently if $y, z \in [x_i, x_{i-1}]$ ⟵ closed interval !! then $|f(y) - f(z)| < r < \frac{\varepsilon}{2} \cdot \frac{1}{b-a}$.

For this partition, we define two step functions f_1 and $f_2 : [a,b] \to \mathbb{R}$ as follows

- If $x = x_i$ $(i = 0, 1, \dots, N)$ is one of the boundary points of the partition, we let
$$f_1(x_i) = f_2(x_i) = f(x_i).$$
- If $x \in (x_{i-1}, x_i)$ then
$$f_1(x) = \text{g.l.b.}\{ f(x) \mid x \in [x_{i-1}, x_i] \}$$
$$f_2(x) = \text{l.u.b}\{ f(x) \mid x \in [x_{i-1}, x_i] \}$$

Then clearly $f_1(x) \le f(x) \le f_2(x)$ for all $x \in [a,b]$.

Moreover, if we pick any $t_i \in (x_{i-1}, x_i)$ then we can calculate $\int_a^b f_2 - f_1$. We find

$$\int_a^b (f_2(x) - f_1(x))\,dx = \sum_{i=1}^{N} (f_2(t) - f_1(t)) \cdot (x_i - x_{i-1})$$

$$\le \sum_{(x_i, x_{i-1}) \subset V} (B - A) \cdot (x_i - x_{i-1})$$ ⟵ contribution from V

$$+ \sum_{(x_i, x_{i-1}) \subset V^c} r \cdot (x_i - x_{i-1})$$ ⟵ from V^c

$$\le (B - A) \sum (x_i - x_{i-1})$$ this is SMALL $< \varepsilon/2$

$$\cdots \sum_{(x_i,x_{i-1})\subset V} \cdots \quad < \frac{\varepsilon/2}{B-A}$$

this is small

$r < \dfrac{\varepsilon/2}{b-a}$

$$+ \; r \cdot \sum_{(x_i,x_{i-1})\subset V^c} (x_i - x_{i-1})$$

$$\leq (B-A)\cdot \frac{\varepsilon/2}{B-A} + \frac{\varepsilon/2}{b-a}\cdot(b-a)$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We have now satisfied the conditions of the proposition on p. 120 in the book, and it follows that f is Riemann integrable.

REMARK 1 The converse is also true:

If $f:[a,b]\to\mathbb{R}$ is Riemann integrable, then the set of points where f is discontinuous has measure zero, and f is bounded.

So now we know exactly which functions have a Riemann integral and which do not.

REMARK 2. All finite sets have measure zero. Therefore all bounded functions that have no more than a finite nr. of discontinuities are Riemann integrable. Step functions are a special case.

But it is also true that all countably infinite sets have measure zero. Here is a proof.

Suppose the set S of points where f is discontinuous is countable. This means that we can label these points by an integer suffix:

$$S = \{a_1, a_2, a_3, a_4, \cdots\}$$

Now fix $\varepsilon > 0$, and let

$$(\alpha_i, \beta_i) = \left(a_i - \frac{\varepsilon}{2^i},\; a_i + \frac{\varepsilon}{2^i}\right)$$

The union of all (α_i, β_i) covers S.

The union of all (α_i, β_i) covers S.
And the total size of these open intervals is

$$\sum_{i=1}^{\infty} (\beta_i - \alpha_i) = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon \left(\sum_{i=1}^{\infty} \frac{1}{2^i} \right) = \varepsilon .$$

$\boxtimes$

It follows that the "ruler function"

$$f : [0,1] \to \mathbb{R}$$

$$f(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ \frac{1}{|q|} & \text{if } x = \frac{p}{q}, \text{ where } p,q \in \mathbb{Z} \\ & \text{and } p,q \text{ have no common divisors other than } 1. \end{cases}$$

is Riemann integrable !!!
The function f is discontinuous only at rational points. The set of rational numbers is countably infinite, and therefore has measure zero.

REMARK. Things can get even worse. There exist sets of measure zero that are UNCOUNTABLY infinite. (for example: the Cantor set), and there are Riemann integrable functions with discontinuities on such a set.
However, there are also functions that are NOT integrable. For example

$$f(x) = \begin{cases} 0, & \text{if } x \text{ irrational} \\ 1, & \text{if } x \text{ rational} \end{cases}$$

$$f : [0,1] \to \mathbb{R}.$$

This function is NOWHERE continuous. And the set $S = [0,1]$ does not have measure zero, so f is NOT Riemann integrable.